Clique Partition Numbers of Boolean Function Graphs $B(K_p, L(G), INC, NINC)$ and $B(K_p, L(G), INC, NINC)$

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Abstract: A clique in a graph $G$ is a complete subgraph of $G$. A clique partition of $G$ is a collection $C$ of cliques such that each edge of $G$ occurs in exactly one clique in $C$. The clique partition number $cp(G)$ is the minimum size of a clique partition of $G$. In this paper upper bounds for the clique partition number of the Boolean function graphs $BF_2(G)$ and $BF_3(G)$ for some standard graphs are obtained.

Keyword: Boolean Function Graph, clique, clique partition.

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph $G$, let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. A clique partition of $G$ is a collection $C$ of cliques such that each edge of $G$ occurs in exactly one clique in $C$. The clique partition number $cp(G)$ is the minimum size of a clique partition of $G$. The Line graphs, Middle graphs, Total graphs and Quasi-total graphs are very much useful in computer networks.

Whitney[16] introduced the concept of the line graph $L(G)$ of a given graph $G$ in 1932. The first characterization of line graph is due to Krausz. The Middle graph $M(G)$ of a graph $G$ was introduced by Hamada and Yoshimura [5]. Characterizations were presented for middle graphs of any graph, tree and complete graphs in [1]. The concept of total graphs was introduced by Behzad [2] in 1966. Sastry and Raju [15] introduced the concept of quasi-total graphs and they solved the graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. This motivates us to define and study other graph operations.

The points and Lines of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. The total graph $T(G)$ of $G$ has vertex set $V(G) \cup E(G)$ and vertices of $T(G)$ are adjacent whenever they are neighbors in $G$. The quasi-total graph [9] $P(G)$ of $G$ is a graph with vertex set as that of $T(G)$ and two vertices are adjacent if and only if they correspond to two nonadjacent vertices of $G$ or to two adjacent edges of $G$ or to a vertex and an edge incident with it in $G$. The middle graph $M(G)$ of $G$ is the one whose vertex set is as that of $T(G)$ and two vertices are adjacent in $M(G)$ whenever either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edges of $G$ incident with it. Clearly, $E(M(G)) = E(T(G)) – E(G)$.
The corona \( G_1 \circ G_2 \) of two graphs \( G_1 \) and \( G_2 \) is the graph obtained by taking one copy of \( G_1 \) of order \( n \) and \( n \) copies of \( G_2 \), and then joining the \( i^{th} \) vertex of \( G_1 \) to every vertex in the \( i^{th} \) copy of \( G_2 \).

For any graph \( G \), let \( V(G) \) and \( E(G) \) denote the vertex set and edge set of \( G \) respectively. The Boolean function graph \( B(K_p, L(G), INC, NINC) \) of \( G \) is a graph with vertex set \( V(G) \cup E(G) \) and two vertices in \( B(K_p, L(G), INC, NINC) \) are adjacent if and only if they correspond to two non-adjacent edges of \( G \) or to a vertex and an edge incident to it in \( G \), or to a vertex and an edge not incident to it in \( G \), where \( L(G) \) is the line graph of \( G \). For brevity, this graph is denoted by \( BF_2(G) \).

For any graph \( G \), let \( V(G) \) and \( E(G) \) denote the vertex set and edge set of \( G \) respectively. The Boolean function graph \( B(K_p, L(G), INC, NINC) \) of \( G \) is a graph with vertex set \( V(G) \cup E(G) \) and two vertices in \( B(K_p, L(G), INC, NINC) \) are adjacent if and only if they correspond to two adjacent edges of \( G \) or to a vertex and an edge incident to it in \( G \), or to a vertex and an edge not incident to it in \( G \), where \( L(G) \) is the line graph of \( G \). For brevity, this graph is denoted by \( BF_3(G) \).

In this paper, upper bounds for the clique partition numbers of the Boolean function graph \( BF_2(G) \) and \( BF_3(G) \) for some standard graphs are obtained. For unexplained terminology and notations, [4] is referred.

2. Clique partition of \( BF_2(G) \)

In the following, clique partition number of path, cycle, star and wheel graphs are found.

**Theorem 2.1:**

For the path \( P_n \) on \( n \) vertices \( (n \geq 6) \), \( cp(BF_2(P_n)) = \begin{cases} \frac{3n^2 + 2n - 1}{4} & \text{if } n \text{ is odd.} \\ \frac{3n^2 + 2n}{4} & \text{if } n \text{ is even.} \end{cases} \)

**Proof:**

Let \( v_1, v_2, v_3, \ldots, v_n \) be the vertices and \( e_1, e_2, \ldots, e_{n-1} \) be the edges of \( P_n \), where \( e_i = (v_i, v_{i+1}) \), \((1 \leq i \leq n - 1)\). Then \( v_1, v_2, v_3, \ldots, v_n, e_1, e_2, \ldots, e_{n-1} \in V(BF_2(P_n)) \), \( |V(BF_2(P_n))| = 2n - 1 \) and \( |E(BF_2(P_n))| = |E(L(P_n))| + n(n - 1) \)

\[
= \frac{(n - 1)(n - 2)}{2} - (n - 2) + n(n - 1)
= \frac{3n^2 - 7n + 6}{2}.
\]

The clique number of \( BF_2(P_n) \) is \( \frac{n}{2} \).
E( BF\_2(P_n) ) = E( \overline{L(P_n)} ) \cup F, \text{ where } F = \bigcup_{i=1}^{n-1} \bigcup_j \{ v_i, e_j \}; |F| = n(n-1).

**Case 1: n is odd**

The edge set of BF\_2(P_n) is decomposed into K_{\frac{n-1}{2}}, K_3 and K_{\frac{\sqrt{n}}{2}}s. Vertex sets of K_{\frac{n-1}{2}} are listed as elements of the sets A_1 and A_2, where A_1 = \{ e_1, e_3, \ldots, e_{n-2} \}; A_2 = \{ e_2, e_4, \ldots, e_{n-1} \}. Vertex sets of K_3s are given by A_3 = \{ (v_i, e_i, e_{i+2}) \}, \text{ for each } i, 1 \leq i \leq \frac{n-3}{2}, \quad < A_1 > \cong < A_2 > \cong \frac{n-3}{2} K_3.

A_4 = \bigcup_{i=1}^{\frac{n-1}{2}} B_i, \text{ where } B_1 = \{ (v_i, e_i, e_{i+3}) \}, i = 2, 3, \ldots, n - 4; B_2 = \{ (v_i, e_i, e_{i+5}) \}, i = 2, 3, \ldots, n - 6; B_3 = \{ (v_i, e_i, e_{i+7}) \}, i = 2, 3, \ldots, n - 8; \ldots, B_{\frac{n-1}{2}} = \{ (v_{n-3}, e_i, e_{i+11}) \}, i = 2, 3 \}\text{ and hence} < A_4 > \cong \frac{n^2 - 8n + 15}{4} K_3.

These cover all the edges of \overline{L(P_n)} and \frac{n^2 - 6n + 9}{2} edges of F. The remaining \frac{n^2 + 4n - 9}{2} edges of F are covered by K_{\frac{\sqrt{n}}{2}}s.

Therefore, BF\_2(P_n) = 2 K_{\frac{n-1}{2}} \bigcup \left( \frac{n^2 - 6n + 9}{4} \right) K_3 \bigcup \left( \frac{n^2 + 4n - 9}{2} \right) K_2 \text{ and hence}

cp(BF\_2(P_n)) = 2 + \left( \frac{n^2 - 6n + 9}{4} \right) + \left( \frac{n^2 + 4n - 9}{2} \right) = \frac{3n^2 + 2n - 1}{4}.

**Case 2: n is even**

The edge set of BF\_2(P_n) is decomposed into K_{\frac{n}{2}}, K_{\frac{n-2}{2}}, K_3 and K_{\frac{\sqrt{n}}{2}}s. Vertex sets of K_{\frac{n}{2}}, K_{\frac{n-2}{2}} are listed as elements of the sets C_1 and C_2. C_1 = \{ e_1, e_3, \ldots, e_{n-1} \}; C_2 = \{ e_2, e_4, \ldots, e_{n-2} \} and < C_1 > \cong \frac{n}{2} K_3, < C_2 > \cong \frac{n-2}{2} K_3. Vertex sets of K_3s are given by C_3 = \bigcup_{i=1}^{\frac{n-3}{2}} D_i, \text{ where } D_1 = \{ (v_i, e_i, e_{i+3}) \}, i = 1, 2, 3, \ldots, n - 4; D_2 = \{ (v_i, e_i, e_{i+5}) \}, i = 1, 2, 3, \ldots, n - 6; \ldots, D_{\frac{n-3}{2}} = \{ (v_{n-4}, e_{i+11}) \}, i = 2, 3 \}. Vertex sets of K_{\frac{\sqrt{n}}{2}}s are given by...
\( D_3 = \{ \{ v_3, e_4, e_7 \}, i = 1, 2, 3, \ldots, n - 8 \}, \ldots \)
\( D_{n-3} = \{ \{ v_{n-4}, e_{n}, e_{(n-3)} \}, i = 1, 2 \} \) and hence
\[ < C_3 > \cong \left( \frac{n^2 - 6n + 8}{4} \right) K_3. \]

These cover all the edges of \( L(P_n) \) and \( \frac{n^2 - 6n + 8}{2} \) edges of \( F \). The remaining \( \frac{n^2 + 4n - 8}{2} \) edges of \( F \) are covered by \( K_2 \)'s.

Therefore, \( BF_2(P_n) = K_n \bigcup K_{\frac{n-2}{2}} \bigcup \left( \frac{n^2 - 6n + 8}{4} \right) K_3 \bigcup \left( \frac{n^2 + 4n - 8}{2} \right) K_2 \) and hence
\[ cp(BF_2(P_n)) = 2 + \left( \frac{n^2 - 6n + 8}{4} \right) + \left( \frac{n^2 + 4n - 8}{2} \right) = \frac{3n^2 + 2n}{4}. \]

Therefore, \( cp(BF_2(P_n)) = \begin{cases} 
\frac{3n^2 + 2n}{4} & \text{if } n \text{ is odd.} \\
\frac{3n^2 + 2n}{4} & \text{if } n \text{ is even.}
\end{cases} \)

\textbf{Theorem 2.2:}

For any cycle \( C_n \) on \( n \) vertices \( (n \geq 6) \), \( cp(BF_2(C_n)) = \begin{cases} 
\frac{3n^2 + 6n - 1}{4} & \text{if } n \text{ is odd.} \\
\frac{3n^2 + 4n + 8}{4} & \text{if } n \text{ is even.}
\end{cases} \)

\textbf{Proof:}

Let \( v_i \) \( (1 \leq i \leq n) \) be the vertices of \( C_n \). Let \( e_i = (v_i, v_{i+1}), (1 \leq i \leq n - 1) \) and \( e_n = (v_n, v_1) \).

Then \( V(BF_2(C_n)) = V(C_n) \bigcup E(C_n), |V(BF_2(C_n))| = 2n, |E(BF_2(C_n))| = |E(L(C_n))| + n^2 = \frac{3n(n-1)}{2}. \) The clique number of \( BF_2(C_n) \) is \( \frac{n+1}{2}. \)

\( E(BF_2(C_n)) = E(L(C_n)) \bigcup F, \) where \( F = \bigcup_{i=1}^{n-1} \left( \bigcup_{j=1}^{i} \{ v_j, e_i \} \right) \) ; \( |F| = n^2. \)

\textbf{Case 1:} \( n \) is odd

The edge set of \( BF_2(C_n) \) is decomposed into \( K_{\frac{n-1}{2}}, K_3 \) and \( K_2 \)'s.

Vertex sets of \( 2 K_{\frac{n-1}{2}} \) are listed as elements of the sets \( A_1 \) and \( A_2 \), where

\( A_1 = \{ e_1, e_{0}, \ldots, e_{n/2} \}, \ A_2 = \{ e_{n/2}, e_{n}, \ldots, e_n \} \), \( < A_1 > \cong < A_2 > \cong K_{\frac{n-1}{2}}. \)
160 Clique Partition Numbers of Boolean Function Graphs $B(K_p, L(G), \text{INC, NINC})$ and $B(K_p, L(G), \text{INC, NINC})$

Vertex sets of $K_3$’s are given by $A_3 = \bigcup_{i=1}^{n-3} B_i$ where $B_1 = \{\{v_i, e_i, e_{i+3}\}, i = 1, 2, 3, \ldots, n - 3\}$, $B_2 = \{\{v_i, e_i, e_{i+5}\}, i = 1, 2, 3, \ldots, n - 5\}$, $B_3 = \{\{v_i, e_i, e_{i+7}\}, i = 1, 2, 3, \ldots, n - 7\}$, $\ldots$, $B_{n-3} = \{\{v_i, e_i, e_{i+(n-2)}\}, i = 1, 2\}$ and hence $<A_3> \cong \left(\frac{n^2 - 4n + 3}{4}\right) K_3$.

These cover $\left(\frac{n^2 - 4n + 3}{2}\right)$ edges of $L(C_n)$ and $\left(\frac{n^2 - 4n + 3}{2}\right)$ edges of $F$. The remaining $\left(\frac{n-3}{2}\right)$ edges of $L(C_n)$ and $\left(\frac{n^2 + 4n - 3}{2}\right)$ edges of $F$ are covered by $K_2$’s and in total there are $\left(\frac{n^2 + 5n - 6}{2}\right)$ $K_2$’s.

Therefore, $BF_2(C_n) = 2 \bigcup_{i=1}^{n-1} \left(\frac{n^2 - 4n + 3}{4}\right) K_3 \bigcup \left(\frac{n^2 + 5n - 6}{2}\right) K_2$ and hence $\text{cp}(BF_2(C_n)) = 2 + \left(\frac{n^2 - 4n + 3}{4}\right) + \left(\frac{n^2 + 5n - 6}{2}\right) = \frac{3n^2 + 6n - 1}{4}$.

Case 2: $n$ is even

The edge set of $BF_2(C_n)$ is decomposed into $K_{\frac{n}{2}}$, $K_3$ and $K_2$’s. Vertex sets of $2K_{\frac{n}{2}}$ are listed as elements of the sets $D_1$ and $D_2$.

$D_1 = \{e_1, e_3, \ldots, e_n\}$; $D_2 = \{e_2, e_4, \ldots, e_n\}$ and $<D_1> \cong <D_2> \cong K_{\frac{n}{2}}$.

Vertex sets of $K_3$’s are given by $D_3 = \{\{v_i, e_i, e_{2i+2}\}, \text{for each } i, 1 \leq i \leq \frac{n-4}{2}\}$ and $<D_1> \cong \frac{n-4}{2} K_3$.

$D_4 = \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} D_{ij}$, where $D_{ij} = \{\{v_i, e_i, e_{i+3}\}, i = 2, 3, \ldots, n - 3\}$, $D_{2j} = \{\{v_i, e_i, e_{i+5}\}, i = 2, 3, \ldots, n - 5\}$, $D_{3j} = \{\{v_i, e_i, e_{i+7}\}, i = 2, 3, \ldots, n - 7\}$, $\ldots$, $D_{n-4j} = \{\{v_{n-j}, e_i, e_i+1\}, i = 2, 3\}$ and $<D_4> \cong \left(\frac{n^2 - 6n + 8}{4}\right)$.

These cover all the edges of $L(C_n)$ and $\left(\frac{n^2 - 4n}{2}\right)$ edges of $F$. The remaining $\left(\frac{n^2 - 4n}{2}\right)$ edges of $F$ are covered by $K_2$’s.
Therefore, $\text{BF}_2(C_n) = 2K_3 \cup \left( \frac{n^2 - 4n}{2} \right) K_3 \cup \left( \frac{n^2 + 4n}{2} \right) K_3$ and hence

$$\text{cp}(\text{BF}_2(C_n)) = 2 + \left( \frac{n^2 - 4n}{2} \right) + \left( \frac{n^2 + 4n}{2} \right) = \frac{3n^2 + 4n + 8}{4}.$$ 

Therefore, $\text{cp}(\text{BF}_2(C_n)) = \begin{cases} 
3n^2 + 6n - 1 & \text{if } n \text{ is odd.} \\
4 & \text{if } n \text{ is even.} 
\end{cases}$

**Theorem 2.3:**

For the star $K_{1,n}$ on $n$ vertices ($n \geq 3$), $\text{cp}(\text{BF}_2(K_{1,n})) = n(n + 1)$.

**Proof:**

Let $v$ be the central vertex and $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $e_1, e_2, \ldots, e_n$ be the edges of $K_{1,n}$, where $e_i = (v, v_i), (1 \leq i \leq n)$.

Then $v, v_1, v_2, v_3, \ldots, v_n, e_1, e_2, \ldots, e_n \in V(\text{BF}_2(K_{1,n}))$ and $|V(\text{BF}_2(K_{1,n}))| = 2n + 1$ and $|E(\text{BF}_2(K_{1,n}))| = n(n + 1)$ and the clique number is 2. Since $\text{BF}_2(K_{1,n})$ is C₃ free and edges of $\text{BF}_2(K_{1,n})$ can be decomposed into $K_2$'s only. The edge sets of $n(n + 1)K_2$ are denoted as $A_1$ and $A_2$ are given as

$$A_1 = \{(v, e_i): 1 \leq i \leq n\} \text{ and } A_2 = \bigcup_{i=1}^{n} \bigcup_{j=i}^{n} \{(v, e_i)\}, \quad |A_1| = n; |A_2| = n^2; |A_1| + |A_2| = n(n + 1).$$

Therefore $\text{cp}(\text{BF}_2(K_{1,n})) = n(n + 1)$.

**Theorem 2.4:**

For the wheel $W_{n+1}$ on $(n+1)$ vertices ($n \geq 6$),

$$\text{cp}(\text{BF}_2(W_{n+1})) = \begin{cases} 
11n^2 - 2n - 1 & \text{if } n \text{ is odd.} \\
4 & \text{if } n \text{ is even.} 
\end{cases}$$

**Proof:**

Let $v$ be the central vertex of $W_{n+1}$ and $v_1, v_2, v_3, \ldots, v_n$ be the vertices of cycle $C_n$. Let $e_i = (v, v_i), 1 \leq i \leq n$ and $j \equiv (i + 1)(\text{mod } n)$ and $f_i = (v, v_i), 1 \leq i \leq n$. Then $V(\text{BF}_2(W_{n+1})) = V(W_{n+1}) \cup E(W_{n+1})$.

$|V(\text{BF}_2(W_{n+1}))| = (n + 1) + (2n) = 3n + 1.$
\[|E(\ BF_2(W_{n+1})|) = |E(L(W_{n+1}))| + 2(n+1)\] and the clique number is \(\frac{n+1}{2}\).

Then \(E(\ BF_2(W_{n+1})|) = E(L(W_{n+1})) \cup F \cup H\), where \(F = \bigcup_{i=1}^{n} \left\{ (v_i, e_i), (v_i, f_i) \right\}\) and \(H = \bigcup_{i=1}^{n} \left\{ (v_i, e_i), (v_i, f_i) \right\}\).

Case 1: \(n\) is odd

The edge set of \(BF_2(W_{n+1})\) is decomposed into \(K_{\frac{n}{2}}, K_3\) and \(K_{2}\)'s.

Vertex sets of \(\frac{n}{2}K_{\frac{n}{2}}\) are listed as elements of the sets \(A_1\) and \(A_2\), where

\[A_1 = \{ e_1, e_3, \ldots, e_{n-2} \};\ A_2 = \{ e_2, e_4, \ldots, e_{n-1} \};\ < A_1 > \cong < A_2 > \cong K_{\frac{n}{2}}.\]

Vertex sets of \(K_3\)'s are given by

\[A_3 = \bigcup_{i=1}^{n} B_i\]

where

\[B_1 = \{ (v_1, e_i, e_{i+3}) \}; i = 1, 2, 3, \ldots, n - 3\].
\[B_2 = \{ (v_2, e_i, e_{i+5}) \}; i = 1, 2, 3, \ldots, n - 5\].
\[B_3 = \{ (v_3, e_i, e_{i+7}) \}; i = 1, 2, 3, \ldots, n - 7\].
\[\ldots\]
\[B_{\frac{n}{2}} = \{ (v_{n-1}, e_i, e_{i+2}) \}; i = 1, 2\] and \(< A_3 > \cong \left( \frac{n^2 - 4n + 3}{4} \right) K_3\).

\[A_4 = \{ (v, f_i, e_{i+1}) \}; 1 \leq i \leq n, e_{n+1} = e_1\] and \(< A_4 > \cong nK_3\).

These cover all the edges of \(H\), \(\frac{n^2 - 2n + 3}{2}\) edges of \(L(W_{n+1})\) and \(\frac{n^2 - 4n + 3}{2}\) edges of \(F\). The remaining \(\frac{5n^2 - n - 6}{2}\) edges are covered by \(K_2\)'s.

Therefore, \(BF_2(W_{n+1}) = 2K_{\frac{n}{2}} \cup \left( \frac{n^2 + 3}{4} \right) K_3 \cup \left( \frac{5n^2 - n - 6}{2} \right) K_2\) and hence

\[cp(BF_2(W_{n+1})) = 2 + \left( \frac{n^2 + 3}{4} \right) + \left( \frac{5n^2 - n - 6}{2} \right) = \frac{11n^2 - 2n - 1}{4}.

Case 2: \(n\) is even

The edge set of \(BF_2(W_{n+1})\) is decomposed into \(K_{\frac{n}{2}}, K_3\) and \(K_2\)'s.
Vertex sets of $2K_n$ are listed as elements of the sets $C_1$ and $C_2$.

$C_1 = \{e_1, e_3, \ldots, e_{n-1}\}$ ; $C_2 = \{e_2, e_4, \ldots, e_n\}$ and $< C_1 > \cong < C_2 > \cong K_{n/2}$.

Vertex sets of $K_n$’s are given by

$C_3 = \{\{v_i, e_1, e_{2i+2}\}, \text{for each } i, 1 \leq i \leq \frac{n-4}{2}\}$ and $< C_3 > \cong \frac{n-4}{2} K_3$.

$C_4 = \bigcup_{i=1}^{\left\lfloor \frac{n}{4} \right\rfloor} D_i$, where $D_1 = \{\{v, e_1, e_{i+2}\}, i = 2, 3, \ldots, n - 3\}$, $D_2 = \{\{v, e_1, e_{i+4}\}, i = 2, 3, \ldots, n - 5\}$, $D_3 = \{\{v, e_1, e_{i+6}\}, i = 2, 3, \ldots, n - 7\}$, $\ldots$, $D_{\left\lfloor \frac{n}{4} \right\rfloor} = \{\{v, e_1, e_{i+(n-3)}\}, i = 2, 3\}$ and $< C_4 > \cong \left(\frac{n^2 - 6n + 8}{4}\right) K_3$.

$c_5 = \{\{v, f_i, e_{i+1}\}; 1 \leq i \leq n, e_{n+1} = e_1\}$ and $< C_5 > \cong nK_3$.

These cover all the edges of $H$, $\left(\frac{n^2 - n}{2}\right)$ edges of $L(C_n)$ and edges $\left(\frac{n^2 - 4n}{2}\right)$ of $F$.

The remaining $\left(\frac{5n^2 - 2n}{2}\right)$ edges of $F$ are covered by $K_2$’s.

Therefore, $BF_c(W_{e+1}) = 2K_{n/2} \bigcup \left(\frac{n^2}{4}\right) K_3 \bigcup \left(\frac{5n^2 - 2n}{2}\right) K_2$ and hence

$cp(BF_c(W_{e+1})) = 2 + \left(\frac{n^2}{4}\right) + \left(\frac{5n^2 - 2n}{2}\right) = \frac{11n^2 - 4n + 8}{4}$.

Therefore, $cp(BF_c(W_{e+1})) = \begin{cases} 
11n^2 - 2n - 1 
& \text{if } n \text{ is odd.} \\
4 
& \text{if } n \text{ is even.} \\
\end{cases}$

In the following clique partition number of $P_n \circ K_i$ and $C_n \circ K_i$ are found.

Theorem 2.5:

For the graph $P_n \circ K_i$ ($n \geq 6$), $cp(BF_c(P_n \circ K_i)) = \begin{cases} 
\frac{9n^2 - 6n + 3}{2} 
& \text{if } n \text{ is odd.} \\
\frac{9n^2 - 7n + 8}{2} 
& \text{if } n \text{ is even.} \\
\end{cases}$
Proof:
Let \( v_i \) (\( 1 \leq i \leq n \)) be the vertices of \( P_n \) with \( v_1 \) and \( v_n \) as pendant vertices and let \( e_i = (v_i, v_{i+1}), \) (\( 1 \leq i \leq n-1 \)) be the edges of \( P_n \). Let \( u_i \) be the pendant vertex adjacent to \( v_i \) (\( 1 \leq i \leq n \)) and let \( f_i = (v_i, u_i), \) (\( 1 \leq i \leq n \)).

\[
V(BF(P_n \circ K_2)) = V(P_n \circ K_2) \cup E(P_n \circ K_2).
\]

Therefore \( |V(BF(P_n \circ K_2))| = 2n + 2n - 1 = 4n - 1. \)

Let \( F = \bigcup_{i=1}^{n-1} \bigcup_{j=1}^{n} \{(v_i, e_i), (u_i, e_i)\} \)

and \( H = \bigcup_{i=1}^{n-1} \bigcup_{j=1}^{n} \{(v_i, f_i), (u_i, f_i)\} \)

\[
|F| = 2n(n - 1); \ |H| = 2n^2. \text{ Then } E(BF(P_n \circ K_2)) = E(L(P_n \circ K_2)) \cup F \cup H.
\]

\[
|E(BF(P_n \circ K_2))| = 6n^2 - 8n + 5. \text{ The clique number of } BF(P_n \circ K_2) \text{ is } \frac{n-1}{2}.
\]

Case 1: \( n \) is odd

Vertex sets of \( K_{\frac{n-1}{2}} \) are listed as elements of the sets \( A_1, A_2, A_3, \) and \( A_4. \)

\( A_1 = \{v_1, e_1, \ldots, e_{\frac{n}{2}}\}; \ A_2 = \{v_2, e_2, \ldots, e_{\frac{n}{2}}\}. \)

\( A_3 = \{f_1, f_3, \ldots, f_{\frac{n}{2}}\}; \ A_4 = \{f_2, f_4, \ldots, f_{\frac{n}{2}}\}. \)

Vertex sets of \( K_3 \)'s are given by

\[
B_i = \bigcup_{j=1}^{\frac{n}{2}} \{v_{\frac{n-1}{2}}, e_{\frac{n}{2}+i}, e_{\frac{n}{2}+i+1}\}, \quad i = 2, 3, \ldots, n-4.
\]

\[
C_i = \bigcup_{j=1}^{\frac{n}{2}} \{u_{\frac{n}{2}+i}, f_{\frac{n}{2}+i}, f_{\frac{n}{2}+i+1}\}, \quad i = 1, 2, 3, \ldots, n-3.
\]

\[
A_6 = \bigcup_{i=1}^{\frac{n}{2}} \{v_{\frac{n}{2}+i}, f_{\frac{n}{2}+i}, f_{\frac{n}{2}+i+1}\}, \quad i = 1, 2, 3, \ldots, n-2.
\]

\[
A_7 = \{v_n, f_{\frac{n}{2}}, f_{\frac{n}{2}+1}\}; \ \text{ and } <A_7> \cong nK_3.
\]
\[ A_8 = \{\{v_i, e_i, e_{n+1}\}, \text{for each } i, 1 \leq i \leq \frac{n-3}{2}\} \text{ and } < A_8 > \cong \frac{n-3}{2} K_3. \]

These cover \( \left( \frac{2n^2 - 7n + 9}{2} \right) \) edges of \( L(P_n \circ K_2) \) and \( \left( \frac{n^2 - 6n + 9}{2} \right) \) edges of \( F \) and

\( \left( \frac{n^2 + 3}{2} \right) \) edges of \( H \). The remaining \( \left( \frac{8n^2 - 3n - 11}{2} \right) \) edges are covered by \( K_2 \)'s.

Therefore, \( BF_2(P \circ K_2) = 4 K_{\frac{n}{2}} \cup \left( \frac{n^2 - 3n + 6}{2} \right) K_3 \cup \left( \frac{8n^2 - 3n - 11}{2} \right) K_2 \) and hence

\[ cp(BF_2(P \circ K_2)) = 4 + \left( \frac{n^2 - 3n + 6}{2} \right) + \left( \frac{8n^2 - 3n - 11}{2} \right) = \frac{9n^2 - 6n + 3}{2}. \]

**Case 2:** \( n \) is even

Vertex sets of \( K_{\frac{n}{2}}, K_{\frac{n-2}{2}} \) are listed as elements of the sets \( D_1, D_2, D_3 \) and \( D_4 \), where

\[ D_1 = \{e_1, e_0, e_{n-1}\}. \]

\[ D_2 = \{e_2, e_0, e_{n-2}\}. \]

\[ D_3 = \{f_1, f_0, f_{n-1}\}. \]

\[ D_4 = \{e_0, e_{n}, e_{n+1}\} \text{ and } < D_1 > \cong < D_3 > \cong < D_4 > \cong K_{\frac{n}{2}} ; < D_2 > \cong K_{\frac{n-2}{2}}. \]

Vertex sets of \( K_3 \)'s are given by

\[ \left\{ \bigcup_{i=1}^{n-4} E_i \right\}, \text{ where} \]

\[ E_1 = \{\{v_1, e_i, e_{i+3}\}, i = 1, 2, 3, \ldots, n-4\}. \]

\[ E_2 = \{\{v_2, e_i, e_{i+5}\}, i = 1, 2, 3, \ldots, n-6\}. \]

\[ E_3 = \{\{v_3, e_i, e_{i+7}\}, i = 1, 2, 3, \ldots, n-8\}. \]

\[ E_{n-4} = \{\{v_{n-4}, e_0, e_{i+(n-3)}\}, i = 1, 2\} \text{ and } < D_5 > \cong \left( \frac{n^2 - 6n + 8}{4} \right) K_3. \]

\[ \left\{ \bigcup_{i=1}^{n-3} J_i \right\}, \text{ where} \]

\[ J_1 = \{\{u_1, f_i, f_{i+3}\}, i = 2, 3, \ldots, n-3\}. \]

\[ J_2 = \{\{u_2, f_i, f_{i+5}\}, i = 2, 3, \ldots, n-5\}. \]

\[ J_3 = \{\{u_3, f_i, f_{i+7}\}, i = 2, 3, \ldots, n-7\}. \]
Clique Partition Numbers of Boolean Function Graphs $\overline{B(K_p,L(G),\text{INC},\text{NINC})}$ and $\overline{B(K_p,L(G),\text{INC},\text{NINC})}$

\[ J_{n-1}^{1} = \{(u_{i-1,n-i}, f_{i}, f_{i+1}) : i = 2, 3\} \text{ and } < D_{n} > \cong \left(\frac{n^2 - 6n + 8}{4}\right) K_3. \]

\[ D_7 = \{(u_i, f_i, f_{i+1}) : 1 \leq i \leq \frac{n-4}{2}\} \text{ and } < D_{7} > \cong \left(\frac{n-4}{2}\right) K_3. \]

\[ D_8 = \{(v_i, f_i, f_{i+1}) : 1 \leq i \leq n, f_{n+1} = f_1\} \text{ and } < D_{8} > \cong nK_3. \]

These cover $n^2 - 3n + 3$ edges of $L(P_n \circ K_1)$ and $\frac{n}{2}$ edges of $F$ and $\left(\frac{n^2 - 2n + 8}{2}\right)$ edges of $H$. The remaining $\left(\frac{8n^2 - 4n - 4}{2}\right)$ edges are covered by $K_3$'s.

Therefore, $BF_2(P_n \circ K_1) = 3K_n \cup \frac{K_{n-2}}{2} \bigcup \left(\frac{n^2 - 3n + 4}{2}\right) K_3 \bigcup \left(\frac{8n^2 - 4n - 4}{2}\right) K_2$

and hence $cp(BF_2(P_n \circ K_1)) = 4 + \left(\frac{n^2 - 3n + 4}{2}\right) + \left(\frac{8n^2 - 4n - 4}{2}\right) = \frac{9n^2 - 7n + 8}{2}.$

Therefore, $cp(BF_2(P_n \circ K_1)) = \begin{cases} 
\frac{9n^2 - 6n + 3}{2} & \text{if } n \text{ is odd,} \\
\frac{9n^2 - 7n + 8}{2} & \text{if } n \text{ is even.}
\end{cases}$

**Theorem 2.6:**

For the graph $C_n \circ K_1 (n \geq 6)$, $cp(BF_2(C_n \circ K_1)) = \begin{cases} 
\frac{9n^2 - 6n + 3}{2} & \text{if } n \text{ is odd,} \\
\frac{9n^2 - 7n + 8}{2} & \text{if } n \text{ is even.}
\end{cases}$

**Proof:**

Let $v_i (1 \leq i \leq n)$ be the vertices of $C_n$ and let $u_i (1 \leq i \leq n)$ be the pendant vertex adjacent to $v_i$. Let $e_i = (v_i, v_{i+1})$, $(1 \leq i \leq n - 1)$, $e_n = (v_n, v_1)$ and $f_i = (v_i, u_i) ; (1 \leq i \leq n)$.

$V(BF_2(C_n \circ K_1)) = V(C_n \circ K_1) \cup E(C_n \circ K_1)$. Therefore $|V(BF_2(C_n \circ K_1))| = 4n.$

Then $E(BF_2(C_n \circ K_1)) = L((C_n \circ K_1)) \cup F,$

where $F = \bigcup_{j=1}^{n} \left\{(v_j, e_j), (v_j, f_j), (u_j, e_j), (u_j, f_j)\right\}; |F| = 4n^3.$

$|E(BF_2(C_n \circ K_1))| = 6n^2 - 4n.$

**Case1:** $n$ is odd
The edge set of $BF_2(C_n \circ K_2)$ is decomposed into $K_{n-1}$, $K_3$ and $K_2$'s.

Vertex set of $K_{n-1}$ are listed as elements of the sets $A_1$, $A_2$, $A_3$ and $A_4$, where

$A_1 = \{e_1, e_3, \ldots, e_{n-2}\}; \quad A_2 = \{e_2, e_4, \ldots, e_{n-1}\}.
A_3 = \{f_1, f_3, \ldots, f_{n-2}\}; \quad A_4 = \{f_2, f_4, \ldots, f_{n-1}\}.

Vertex sets of $K_3$'s are given by

$A_5 = \bigsqcup_{i=1}^{n-3} B_i$, where

$B_1 = \{\{v_i, e_i, e_{i+3}\}, i = 1, 2, 3, \ldots, n-3\}.$
$B_2 = \{\{v_2, e_i, e_{i+5}\}, i = 1, 2, 3, \ldots, n-5\}.$
$B_3 = \{\{v_3, e_i, e_{i+7}\}, i = 2, 3, \ldots, n-7\}.$

$A_6 = \bigsqcup_{i=1}^{n-3} C_i$, where

$C_1 = \{\{u_1, f_i, f_{i+3}\}, i = 1, 2, 3, \ldots, n-3\}.$
$C_2 = \{\{u_2, f_i, f_{i+5}\}, i = 1, 2, 3, \ldots, n-5\}.$
$C_3 = \{\{u_3, f_i, f_{i+7}\}, i = 1, 2, 3, \ldots, n-7\}.$

$A_7 = \{\{v_i, f_i, f_{i+1}\}; 1 \leq i \leq n, f_{n+1} = f_1\}$ and $\langle A_7 \rangle \cong 2K_3.$

These cover $n^2 - 3n + 3$ edges of $L(C_n \circ K_2)$ and $n^2 - 2n + 3$ edges of $F$. The remaining $4n^2 + n - 6$ edges are covered by $K_2$'s.

Therefore, $BF_2(C_n \circ K_2) = 4K_{n-1} \cup \left(\frac{n^2 - 2n + 3}{2}\right)K_3 \cup (4n^2 + n - 6)K_2$ and hence

$cp(BF_2(C_n \circ K_2)) = 4 + \left(\frac{n^2 - 2n + 3}{2}\right) + (4n^2 + n - 6) = \frac{9n^2 - 1}{2}.$

Case 2: $n$ is even

The edge set of $BF_2(C_n \circ K_2)$ is decomposed into edges of $K_{n-1}$, $K_3$, $K_2$'s.
Vertex sets of 4 $K_n$ are listed as elements of the sets $D_1$, $D_2$, $D_3$ and $D_4$, where

$D_1 = \{e_1, e_3, \ldots, e_{n-1}\}$; \quad $D_2 = \{e_2, e_4, \ldots, e_n\}$.

$D_3 = \{f_1, f_3, \ldots, f_{n-1}\}$; \quad $D_4 = \{f_2, f_4, \ldots, f_n\}$.

Vertex sets of $K_{3, s}'$s are given by

$D_5 = \{\{v_i, e_1, e_{2i+2}\}; 1 \leq i \leq \frac{n-4}{2}\}$ and $< D_5 > \cong \left(\frac{n-4}{2}\right) K_3$.

$D_6 = \left(\frac{n-4}{2}\right) E_i$ where

$E_i = \{\{v_i, e_0, e_{i+3}\}; i = 2, 3, \ldots, n-3\}$.

$E_2 = \{\{v_2, e_0, e_{i+5}\}; i = 2, 3, \ldots, n-5\}$.

$E_3 = \{\{v_3, e_0, e_{i+7}\}; i = 2, 3, \ldots, n-7\}$.

$E_{n-4} = \left(\frac{n-4}{2}\right) J_i$ and $< D_6 > \cong \left(\frac{n^2 - 6n + 8}{4}\right) K_3$.

$D_7 = \left(\frac{n-4}{2}\right) J_i$ where

$J_i = \{\{u_i, f_0, f_{i+3}\}; i = 2, 3, \ldots, n-3\}$.

$J_2 = \{\{u_2, f_0, f_{i+5}\}; i = 2, 3, \ldots, n-5\}$.

$J_3 = \{\{u_3, f_0, f_{i+7}\}; i = 2, 3, \ldots, n-7\}$.

$J_{n-4} = \left(\frac{n-4}{2}\right) J_i$ and $< D_7 > \cong \left(\frac{n^2 - 6n + 8}{4}\right) K_3$.

$D_8 = \{\{u_i, f_0, f_{2i+2}\}; 1 \leq i \leq \frac{n-4}{2}\}$ and $< D_8 > \cong \frac{n-4}{2} K_3$.

$D_9 = \{\{v_i, f_0, f_{i+1}\}; 1 \leq i \leq n, f_{n+1} = f_1\}$ and $< D_9 > \cong nK_3$.

These cover $n^2 - 2n$ edges of $L(C_n \circ K_2)$ and $n^2 - 2n$ edges of $F$.

The remaining $4n^2$ edges are covered by $K_{3, s}'$s.

Therefore, $BF_2(C_n \circ K_2) = 4 K_3 \bigcup \left(\frac{n^2 - 2n}{2}\right) K_3 \bigcup (4n^2) K_2$ and hence
\[ \text{cp}(B_{j}^{*}(C_{n} \circ K_{j})) = 4 + \left( \frac{n^2 - 2n}{2} \right) + 4n^2 = \frac{9n^2 - 2n + 4}{2}. \]

Therefore, \[ \text{cp}(B_{j}^{*}(C_{n} \circ K_{j})) = \begin{cases} 
\frac{9n^2 - 1}{2} & \text{if } n \text{ is odd,} \\
\frac{9n^2 - 2n + 4}{2} & \text{if } n \text{ is even.}
\end{cases} \]

3. **Clique partition of BF_{3}(G)**

In the following, clique partition number of path, cycle, star and wheel graphs are found.

**Theorem 3.1:**
For the path \( P_{n} \) on \( n \) vertices \( (n \geq 5) \), \( \text{cp}(B_{j}^{*}(P_{n})) = n^2 - 2n + 2. \)

**Proof:**
Let \( v_{1}, v_{2}, v_{3}, \ldots, v_{n} \) be the vertices and \( e_{1}, e_{2}, \ldots, e_{n-1} \) be the edges of \( P_{n} \), where \( e_{i} = (v_{i}, v_{i+1}), (1 \leq i \leq n - 1) \). Then \( v_{1}, v_{2}, v_{3}, \ldots, v_{n}, e_{1}, e_{2}, \ldots, e_{n-1} \in V(B_{j}^{*}(P_{n})) \) and \( |V(B_{j}^{*}(P_{n}))| = 2n - 1, |E(B_{j}^{*}(P_{n}))| = |E(L(P_{n}))| + n(n - 2) = n^2 - 2n + 2. \)

The clique number of \( B_{j}^{*}(P_{n}) \) is 3.

\[ E(B_{j}^{*}(P_{n})) = E(L(P_{n})) \bigcup F, \text{ where } F = \bigcup_{j=1}^{n-1} \left( \bigcup_{i=1}^{j} e_{i} \right); |F| = n(n - 1). \]

The edge set of \( B_{j}^{*}(P_{n}) \) is decomposed into \( K_{3} \) and \( K_{2}'s. \)

Vertex sets of \( K_{2}'s \) is given by \( B = \{e_{j}, e_{j+1}, e_{j+2}\}, \) for each \( i, 1 \leq i \leq n - 2\).

These cover all the edges of \( L(P_{n}) \) and \( 2(n - 2) \) edges of \( F. \) The remaining \( (n^2 - 3n + 4) \) edges in \( F \) are covered by \( K_{2}'s. \)

Therefore, \( B_{j}^{*}(P_{n}) = (n - 2)K_{3} \bigcup (n^2 - 3n + 4)K_{2} \) and hence \( \text{cp}(B_{j}^{*}(P_{n})) = n^2 - 2n + 2. \)

**Theorem 3.2:**
For the cycle \( C_{n} \) on \( n \) vertices \( (n \geq 5) \), \( \text{cp}(B_{j}^{*}(C_{n})) = n^2 - n. \)

**Proof:**
Let \( v_{1}, v_{2}, v_{3}, \ldots, v_{n} \) be the vertices and \( e_{1}, e_{2}, \ldots, e_{n} \) be the edges of \( C_{n} \), where \( e_{i} = (v_{i}, v_{i+1}), (1 \leq i \leq n - 1) \) and \( e_{n} = (v_{n}, v_{1}). \)

\[ V(B_{j}^{*}(C_{n})) = V(C_{n}) \bigcup E(C_{n}). \text{ Then } |V(B_{j}^{*}(C_{n}))| = 2n \text{ and } |E(B_{j}^{*}(C_{n}))| = |E(L(C_{n}))| + n = n^2 + n. \text{ The clique number of } B_{j}^{*}(C_{n}) \text{ is 3.} \]

\[ E(B_{j}^{*}(C_{n})) = E(L(C_{n})) \bigcup F, \text{ where } F = \bigcup_{j=1}^{n} \left( \bigcup_{i=1}^{j} e_{i} \right); |F| = n^2. \]
The edge set of $BF_3(C_n)$ is decomposed into $K_3$ and $K_2$'s.

Vertex sets of $K_3$'s is given by

$$C = \{ (e_i, e_{i+1}, v_{i+1}), \text{ for each } i, 1 \leq i \leq n \}, \quad v_{n+1} = v_1, \quad e_0 = e_1.$$

These sets cover all the edges of $L(C_n)$ and $2n$ edges of $F$. The remaining $n(n - 2)$ edges are covered by $K_2$'s. Therefore $BF_3(C_n) = nK_3 \cup (n(n - 2))K_2$ and hence $cp(BF_3(C_n)) = n + n(n - 2) = n^2 - n$.

**Theorem 3.3:**

For the star $K_{1,n}$ on $n$ vertices ($n \geq 6$), $cp(BF_3(K_{1,n})) = \begin{cases} \frac{3n^2 + 14n - 1}{4} & \text{if } n \text{ is odd.} \\ \frac{3n^2 + 12n + 8}{4} & \text{if } n \text{ is even.} \end{cases}$

**Proof:**

Let $v$ be the central vertex and $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $e_1, e_2, \ldots, e_n$ be the edges of $K_{1,n}$, where $e_i = (v, v_i)$, $(1 \leq i \leq n)$.

Then $v, v_1, v_2, v_3, \ldots, v_n, e_1, e_2, \ldots, e_n = V(BF_3(K_{1,n}))$ and $|V(BF_3(K_{1,n}))| = 2n + 1$ and

$$|E(BF_3(K_{1,n}))| = E(L(K_{1,n})) + n(n + 1) = \frac{n(3n + 1)}{2}$$

and the clique number is $\frac{n}{2}$.

$$E(BF_3(K_{1,n})) = E(L(BF_3(K_{1,n}))) \cup F \cup H,$$

where

$$F = \{(v, e_i): 1 \leq i \leq n\}; \quad H = \bigcup_{j=1}^{n} \bigcup_{j=1}^{n} (v_j, e_i).$$

$|F| = n$ and $|H| = n^2$.

**Case 1:** $n$ is odd.

The edge set of $BF_3(K_{1,n})$ is decomposed into $K_n, K_3$, and $K_2$'s.

Vertex sets of $2K_{n-1}$ are listed as elements of the sets $A_1$ and $A_2$, where

$$A_1 = \{ e_1, e_2, \ldots, e_n \} \quad A_2 = \{ e_2, e_3, \ldots, e_{n-1} \}.$$

Vertex sets of $K_2$'s are given by

$$A_3 = \bigcup_{i=1}^{n-1} B_i, \text{ where }$$

$$B_1 = \{ (v_1, e_0, e_{i+3}), i = 1, 2, 3, \ldots, n - 3 \}. \quad B_2 = \{ (v_2, e_0, e_{i+5}), i = 1, 2, 3, \ldots, n - 5 \}. \quad B_3 = \{ (v_3, e_0, e_{i+7}), i = 1, 2, 3, \ldots, n - 7 \}. $$

...
\[ B_{n-4} = \{ \{ v_{n-4}, e_0, e_{i_1(n-2)} \}, i = 1, 2 \} \text{ and } < A_3 > \cong \frac{n^2 - 4n + 3}{4} K_3. \]

These cover \( \left( \frac{n^2 - 4n + 3}{2} \right) \) edges of \( L(K_{1,n}) \) and \( H \). The remaining \( \left( \frac{n^2 + 9n - 6}{2} \right) \) edges are covered by \( K_2 \)'s.

Therefore, \( BF_3(K_{1,n}) = 2 K_{n-1} \bigcup \left( \frac{n^2 - 4n + 3}{4} \right) K_3 \bigcup \left( \frac{n^2 + 9n - 6}{2} \right) K_3 \) and hence

\[ cp(BF_3(K_{1,n})) = 2 + \left( \frac{n^2 - 4n + 3}{4} \right) + \left( \frac{n^2 + 9n - 6}{2} \right) = \frac{3n^2 + 14n - 1}{4}. \]

**Case 2:** \( n \) is even

The edge set of \( BF_3(K_{1,n}) \) is decomposed into \( K_n, K_3 \) and \( K_2 \)'s.

Vertex sets of \( 2 K_n \) are listed as elements of the sets \( C_1 \) and \( C_2 \).

\[ C_1 = \{ e_1, e_2, \ldots, e_n \}; \quad C_2 = \{ e_2, e_4, \ldots, e_n \}. \]

Vertex sets of \( K_3 \)'s are given by

\[ C_3 = \{(v_i, e_0, e_{i+3}) \text{ for each } i, 1 \leq i \leq \frac{n-4}{2} \} \text{ and } < C_3 > \cong \frac{n-4}{2} K_3 \]

\[ C_4 = \bigcup_{i=1}^{\frac{n-4}{2}} D_i \text{ where } \]

\[ D_1 = \{ \{ v_i, e_0, e_{i+3} \}, i = 2, 3, \ldots, n - 3 \}. \]

\[ D_2 = \{ \{ v_0, e_0, e_{i+3} \}, i = 2, 3, \ldots, n - 5 \}. \]

\[ D_3 = \{ \{ v_3, e_0, e_{i+7} \}, i = 2, 3, \ldots, n - 7 \}. \]

\[ \ldots \]

\[ D_{n-4} = \{ \{ v_{n-4}, e_0, e_{i_1(n-3)} \}, i = 2, 3 \} \text{ and } < C_4 > \cong \left( \frac{n^2 - 6n + 8}{4} \right) K_3. \]

These cover \( \left( \frac{n^2 - 3n}{2} \right) \) edges of \( L(K_{1,n}) \) and edges \( \left( \frac{n^2 - 4n}{2} \right) \) of \( H \). The remaining \( \left( \frac{n^2 + 8n}{2} \right) \) edges are covered by \( K_2 \)'s.
Therefore, $BF_3(K_{1, n}) = 2K_n \cup \left( \frac{n^2 - 4n}{4} \right) K_4 \cup \left( \frac{n^2 + 8n}{2} \right) K_3$ and hence

$$cp(BF_3(K_{1, n})) = 2 + \left( \frac{n^2 - 4n}{4} \right) + \left( \frac{n^2 + 8n}{2} \right) = \frac{3n^2 + 12n + 8}{4}.$$ 

Therefore, $cp(BF_3(K_{1, n})) = \begin{cases} 
3n^2 + 14n - 1 & \text{if } n \text{ is odd}; \\
4 & \text{if } n \text{ is even.}
\end{cases}$

**Theorem 3.4:**

For the wheel $W_{n+1}$ on $(n+1)$ vertices $(n \geq 6)$, $cp(3n + 1BF(W_{n+1})) = 2n^2 - n + 1$.

**Proof:**

Let $v$ be the central vertex of $W_{n+1}$ and $v_1, v_2, v_3, \ldots, v_n$ be the vertices of cycle $C_n$. Let $e_i = (v_i, v_j), 1 \leq i \leq n$ and $j \equiv (i + 1) \text{ (mod n)}$ and $f_i = (v_i, v_j), 1 \leq i \leq n$.

Then $V(BF_3(W_{n+1})) = V(W_{n+1}) \cup E(W_{n+1}), |V(BF_3(W_{n+1}))| = (n + 1) + (2n) = 3n + 1$.

$|E(BF_3(W_{n+1}))| = |E(L(W_{n+1}))| + 2n (n + 1) = 3n + \frac{n(n - 1)}{2} + 2n(n + 1) = \frac{n(5n + 9)}{2}$ and the clique number of $W_{n+1}$ is $n$.

Then $|E(BF_3(W_{n+1}))| = |E(L(W_{n+1}))| \cup |E(K_n)| \cup F \cup H$, where

$F = \bigcup_{i=1}^{n} \{(v_i, e_i), (v_i, f_i)\}; |F| = 2n$ and $H = \bigcup_{i=1}^{n} \{(v_i, e_i), (v_i, f_i)\}; |H| = 2n^2$.

The edge set of $BF_3(W_{n+1})$ is decomposed into $K_n, K_3$ and $K_2$'s.

$V(K_n) = \{f_1, f_2, \ldots, f_n\}$;

Vertex sets of $K_3$'s are given by

$B_1 = \{(e_i, e_{i+1}, v_i), 1 \leq i \leq n\}$ and

$B_2 = \{(e_i, f_{i+1}, v_{i+1}), 1 \leq i \leq n, v_{n+1} = v_1, f_{n+1} = f_1\}$ and

$B_3 = \{(e_i, f_{i+1}, v_{i+3}), 1 \leq i \leq n, e_{n+1} = e_1, v_{n+1} = v_3, f_{n+1} = f_1\}$ and

$< B_1 > \cong < B_2 > \cong < B_3 > \cong nK_3$.

The sets $V(K_n), B_1, B_2$ and $B_3$ cover all the edges of $K_n, L(W_{n+1})$ and $6n$ edges of $F$. The remaining $2n^2 - 4n$ edges are covered by $K_2$'s. Therefore $BF_3(W_{n+1}) = K_n \cup (3n)K_3 \cup (2n^2 - 4n)K_2$ and hence $cp(BF_3(W_{n+1})) = 1 + 3n + 2n^2 - 4n = 2n^2 - n + 1$.

In the following, clique partition number of $p \circ K_1$ and $C_n \circ K_1$ are found.
**Theorem 3.5:**
For the graph \( P_n \circ K_1 \) (\( n \geq 6 \)), \( cp(BF_j(P_n \circ K_1)) = 4n^2 - 5n + 4 \).

**Proof:**
Let \( v_i \) (\( 1 \leq i \leq n \)) be the vertices of \( P_n \) with \( v_1 \) and \( v_n \) as pendant vertices and let \( e_i = (v_i, v_{i+1}), (1 \leq i \leq n-1) \) be the edges of \( P_n \). Let \( u_i \) be the pendant vertex adjacent to \( v_i \) (\( 1 \leq i \leq n \)) and let \( f_i = (v_i, u_i), (1 \leq i \leq n) \).

\[ V(BF_j(P_n \circ K_1)) = V(P_n \circ K_1) \cup E(P_n \circ K_1). \]

Therefore \( |V(BF_j(P_n \circ K_1))| = 2n + 2n - 1 = 4n - 1 \).

Let \( F = \bigcup_{j=1}^{n} \left( \bigcup_{i=1}^{n} (v_i, f_i), (u_i, f_i) \right) \) and \( H = \bigcup_{j=1}^{n} \left( \bigcup_{i=1}^{n} (v_i, e_i), (u_i, e_i) \right) \). Then \( |F| = 2n^2 \) and \( |H| = 2n(n-1) \).

\[ |E(BF_j(P_n \circ K_1))| = |E(L(P_n \circ K_1))| + 2n(2n - 1) = (2n - 1)(2n - 1) + \frac{(9n - 10)}{2} = 4n^2 + n - 4. \]

The clique number of \( BF_j(P_n \circ K_1) \) is 3.

The edge set of \( BF_j(P_n \circ K_1) \) is decomposed into \( K_3 \) and \( K_2 \)'s.

Vertex sets of \( K_2 \)'s are given by
\[ C_1 = \{(e_i, e_{i+1}, v_i), 1 \leq i \leq n - 2 \}, \]
\[ C_2 = \{(e_i, f_{i+1}, v_i), 1 \leq i \leq n - 1 \} \] and
\[ C_3 = \{(e_i, u_i, f_{i+1}), \text{ for each } i, 1 \leq i \leq n - 1\} \text{ and } < C_1 > \cong (n - 2)K_3, \]
\[ < C_2 > \cong (n - 1)K_3. \]

The sets \( C_1 \), \( C_2 \) and \( C_3 \) cover all the edges of \( L(P_n \circ K_1) \). \( 2(n - 2) \) edges of \( H \). \( (n - 1) \) edges of \( F \) are covered by \( C_1 \) and \( C_2 \) respectively. \( 2(n - 1) \) edges of \( H \) and \( (n - 1) \) edges of \( F \) are covered both by \( C_2 \) and \( C_3 \). The remaining \( (4n^2 - 8n + 8) \) edges are covered by \( K_2 \)'s.

Therefore \( BF_j(P_n \circ K_1) = (3n - 4)K_3 \bigcup (4n^2 - 8n + 8)K_2 \) and hence
\[ cp(BF_j(P_n \circ K_1)) = 3n - 4 + 4n^2 - 8n + 8 = 4n^2 - 5n + 4. \]

**Theorem 3.6:**
For the graph \( C_n \circ K_1 \) (\( n \geq 6 \)), \( cp(BF_j(C_n \circ K_1)) = 4n^2 - 3n \).

**Proof:**
Let \( v_i \) (\( 1 \leq i \leq n \)) be the vertices of \( C_n \) and let \( u_i \) (\( 1 \leq i \leq n \)) be the pendant vertex adjacent to \( v_i \). Let \( e_i = (v_i, v_{i+1}), (1 \leq i \leq n - 1), e_n = (v_n, v_1) \) and \( f_i = (v_i, u_i), (1 \leq i \leq n) \).

\[ V(BF_j(C_n \circ K_1)) = V(C_n \circ K_1) \cup E(C_n \circ K_1). \]

Therefore \( |V(BF_j(C_n \circ K_1))| = 2n + 2n = 4n. \)

Let \( F = \bigcup_{j=1}^{n} \left( \bigcup_{i=1}^{n} (v_i, e_i), (v_i, f_i), (u_i, e_i), (u_i, f_i) \right) \). Then \( E(BF_j(C_n \circ K_1)) = E(L(C_n \circ K_1)) \bigcup F. \)

\[ |E(BF_j(C_n \circ K_1))| = \frac{8n^2 + 6n}{2} = n(4n + 3). \]
The clique number of $B(F_1(C_n \circ K))$ is 3.

Edge set of $B(F_1(C_n \circ K))$ is decomposed into $K_3$ and $K_2$ s.

Vertex sets of $K_3$ s are given by

$B_1 = \{\{e_i, e_{i+1}, v_i\}, \text{ for each } i, 1 \leq i \leq n, e_{n+1} = e_1\}$ and 

$B_2 = \{\{e_i, v_{i+1}, f_{i+1}\}, \text{ for each } i, 1 \leq i \leq n, f_{n+1} = f_1\}$.

$B_3 = \{\{e_{i+1}, u_i, f_{i+1}\}, \text{ for each } i, 1 \leq i \leq n, e_{n+1} = e_1, f_{n+1} = f_1\}$ and  

$\langle B_1 \rangle \cong \langle B_2 \rangle \cong \langle B_3 \rangle \cong nK_3$.

The sets $B_1$, $B_2$ and $B_3$ cover all the edges of $L(C_n \circ K)$ and 6n edges of F. The remaining 

$n(4n – 6)$ edges of F are covered by $K_2$ s.

Therefore, $B(F_1(C_n \circ K)) = (3n)K_3 \bigcup (4n^2 – 6n) K_2$ and hence 

$cp(B(F_1(C_n \circ K))) = 3n + 4n^2 – 6n = 4n^2 – 3n$.

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